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On the number of mutually touching cylinders. Is it 8?

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Abstract

We solve a problem of W. Kuperberg, who designed an intricate arrangement of eight cylinders and asked if among them there are two which do not have a common point.

1. Introduction

The following problem was posed by Littlewood [3]. What is the maximum number of congruent infinite cylinders that can be arranged in 3-space so that any two of them are touching? Is it 7?

This problem is indeed very interesting and surprisingly hard. Bezdek [1] proved that this number cannot exceed 24. There are several possible types of arrangements of six mutually touching infinite cylinders (see the survey of Brass, Pach and Moser [2]), and the ones we know are flexible with one degree of freedom, hence suggesting a possibility for extending the configuration with a seventh cylinder.

In the early 1990's, W. Kuperberg assembled a nice, symmetrical arrangement of eight pencils (see Fig. 1), showed the physical model to people (i.e. he did not specify all the parameters), and asked them to decide whether the pencils are mutually touching or not. It was such a close call, that people wondered if for specific choices of parameters the pencils can indeed be mutually touching. The purpose of this paper is to show that this is not the case. We also want to show an approach, which could lead to better bounds of the original problem of Littlewood.

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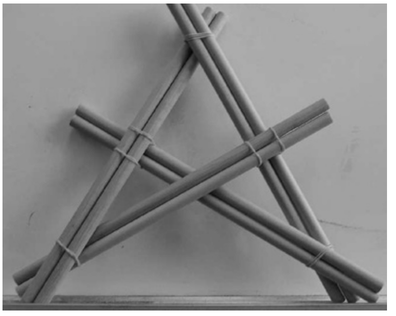


Fig. 1.

Theorem 1. *Among the eight pencils (see Fig. 1) of the configuration of W. Kuperberg, there are two not sharing a common point.*

Theorem 1 is unusual in the sense that it is a precise mathematical statement about a model which itself is not described mathematically. How will we handle this obstacle rigorously? First we find and state couple of properties which are undoubtedly satisfied by the 8 pencil model, and based on them, we prove some lemmas. In this paper the term *cylinder* always refers to an infinite circular cylinder in 3-space. As always we say that: (i) a cylinder **touches** a plane, if their intersection is a line, (ii) two cylinders **are touching each other**, if their intersection is either a single point or a line, (iii) a cylinder **is parallel** to a plane H, if its axis is parallel to the plane. Notice that for two touching cylinders, their separating plane is unique. Let us continue with two intuitive definitions.

The following definition will establish the order between two cylinders with respect to a given direction $\mathbf{u}$. We want to describe in a precise manner, which cylinder do we encounter first as we move from negative infinity to positive infinity along the direction of $\mathbf{u}$.

Definition 1. Consider two cylinders c_1, c_2 and a direction vector $\mathbf{u}$. We say that c_1 is **in front of** c_2 in direction $\mathbf{u}$, if the axis of c_1 can be translated in the direction of $\mathbf{u}$ to infinity without crossing the axis of c_2.

Although Definition 1 is designed to compare disjoint cylinders, notice that it also includes the case of overlapping cylinders. By no confusion, we will also use the "in front" terminology with a directed line in place of the direction vector. Notice also that if a plane separates two cylinders then for any direction vector in the plane, both of the cylinders is in front of the other.

Definition 2. Assume that two cylinders c_1 and c_2 are parallel to a uniquely defined plane H. Let $\mathbf{u}$ be a normal direction to H such that c_1 lies in front of c_2 in this direction. We say that c_2 is **rotated positively** with respect to c_1, if a positive rotation by an angle $0 < \alpha < \frac{\pi}{2}$ around a directed axis in direction of $\mathbf{u}$ takes c_1 in a position parallel to c_2 (a rotation is positive if it is consistent with the right-hand rule along a directed axis in the direction of $\mathbf{u}$).

Although Definition 2 makes sense for overlapping cylinders too, it is important to visualize its meaning in case of touching cylinders: c_2 is rotated positively with respect to a touching cylinder c_1, if looking down to the unique separating plane of the cylinders from c_1 towards c_2, a counterclockwise rotation takes c_1 to a position parallel to c_2 (see Fig. 2).

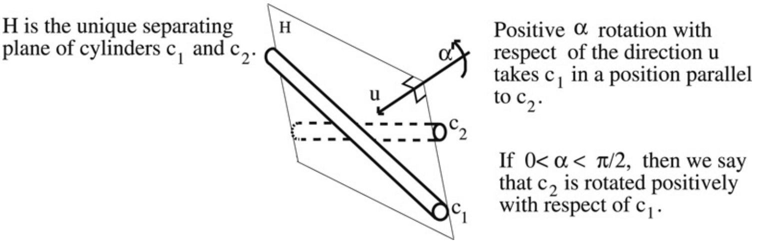


Fig. 2.

Next we list important properties of the pencil model:

Observation 1. *The pencils are considered as congruent infinite circular cylinders.*

The model shows only a specific portion of the infinite cylinders.

The eight cylinders of the configuration are arranged in four 'pairs' (see Figs. 1 and 3). In each pair, the two cylinders seem to be parallel, but it is easy to see that they have to be nonparallel. Just note that, if two cylinders are touching each other along a line, then their axis span a plane and that immediately limits the size of the mutually touching family of cylinders to four. The planes separating the cylinders belonging to a pair will be called *vertical-type planes*. The planes separating the cylinders of the remaining pairs will be called *horizontal-type planes*. The following observations are simple facts of visual study.

Observation 2. *The angles between horizontal-type and vertical-type planes seem to be close to* 90°*, but we will not need more than the fact that they are all larger than* 60°.

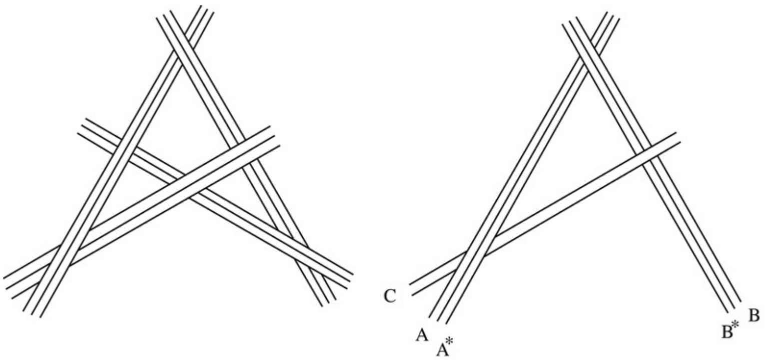
This follows from the fact that otherwise the tips of pencils forming a pair would not appear to be very close to each other. We will not use the value of 60° for computations, our aim is to exclude the extreme situations and say that our figures depicting subsets of the cylinders are realistic.

Theorem 1 is an immediate consequence of the following

Theorem 2. *Among the five pencil subarrangement shown on Fig. 3 there must be two pencils which do not touch each other.*

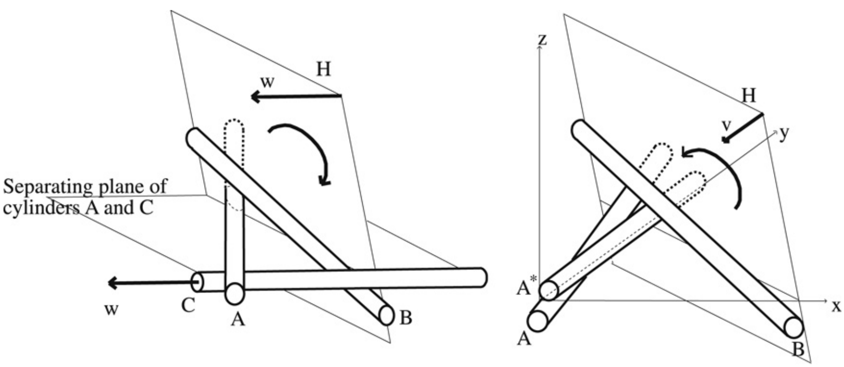
Proof. First we give an outline for the proof of Theorem 2:

Let us label the five cylinders in Theorem 2 by A, A^*, B, B^* and C according to Fig. 3. Assume that the five cylinders are mutually touching. Notice that A, A^* and B, B^* are pairs of almost parallel cylinders with unique separating planes. First we will prove that B^* is rotated positively with respect to B. We will reach a contradiction by proving the opposite too, i.e. it will turn out that B is rotated positively with respect to B^*. To achieve this, we will analyze two different subconfigurations: the first is the family of the cylinders A, C, B and B^*, while the second is the family of the cylinders A, A^*, B and B^*. We will look at these cylinders (and draw conclusions) in the following way: We are going to fix the first two of the four given cylinders, then we take the third cylinder and move it (pretending that the fourth cylinder is not there to block the motion) continuously to the position of the fourth cylinder so that it maintains contact with the two fixed cylinders and remains parallel to the plane which initially separated it from its "pair". According to Lemmas 1 and 2 the moving cylinders rotate in a specific direction. As we said the directions of the rotations will provide the needed contradiction.



W. Kuperberg's model. A subset of W. Kuperberg's model, where the cylinders can not be mutually touching.

Fig. 3.



Lemma 1 Lemma 2

Fig. 4.

Next we give details of the proof of Theorem 2:

For Lemma 1 we fix the non-parallel cylinders A and C. Cylinder B is going to be the cylinder which we move. As we see on Fig. 4, at the beginning B is touching A and C in an interlocking position. Assign the direction $\mathbf{w}$ to the axis of C such that A lies in front of B. Take a plane H such that B is in front of H in direction $\mathbf{w}$, B touches H, and H forms at least 60° with the separating plane of A and C.

Lemma 1. *Translate H along $\mathbf{w}$ to the plane H', and move continuously the cylinder B always requiring to touch A, C and the moving plane H. Denote B' the terminal position, i.e. B' is a cylinder which touches A, C and H'. Then B' is rotated positively with respect to B.*

Proof. Since A lies in front of B in the direction $\mathbf{w}$, if we translate B towards $\mathbf{w}$, it will touch C and intersect A. Therefore in order to be in touching position, the section of B intersecting A 'must be lifted', meaning that a clockwise rotation is applied to the cylinder B. $\square$

Applying Lemma 1 first to the fixed cylinders A and C and the moving cylinder B so that H is a plane parallel to the separating plane of B and B^* we conclude that

Corollary 1. *Cylinder B^* is rotated positively with respect to B.*

Applying Lemma 1 so that: (i) cylinders A and C play the role of the fixed cylinders B and C and (ii) A plays the role of the moving cylinder B and (iii) H is a plane parallel to the separating plane of A and A^*, we conclude that

Corollary 2. *Cylinder A^* is rotated positively with respect to A.*

For Lemma 2, we fix A and A^*. Cylinder B is going to be the cylinder which we move. As we see on Fig. 4 at the beginning these three cylinders are not in interlocking position. To orient ourselves, we choose a coordinate system so that A^* touches the coordinate planes xy and yz, and A^* lies in front of A with respect to the x direction. Assign the direction $\mathbf{v}$ to the axis of A^* such that B lies in front of A with respect $\mathbf{v}$. Let H be a plane which touches cylinder B, forms at least 60° with the coordinate plane xy, and B is in front of H in the direction $\mathbf{v}$.

Lemma 2. *Let us translate the plane H in the direction of $\mathbf{v}$ to the plane H', and move continuously the cylinder B maintaining contact with A, C and with the moving plane H. Denote B' the terminal position, i.e. B' is a cylinder which touches A, A^* and H'. Then B' is rotated positively relative to B.*

Proof. If we translate B towards $\mathbf{v}$ then it touches A^* and does not contact A. Hence the point of B closest to A 'must be lowered', that is, a counterclockwise rotation is applied relative to H. □

Applying Lemma 2 to the fixed cylinders A and A^* and to the moving cylinder B so that H is a plane parallel to the separating plane of B and B^* we conclude that

Corollary 3. *Cylinder B is rotated positively with respect to B^*.*

Finally notice that Corollaries 1 and 3 contradict each other, and thus they prove Theorem 2. □

Although the proof is designed for this special construction, it also shows that configurations by 'doubling cylinders' cannot work. Therefore it strongly restricts the possible combinatorial types of models with many mutually touching cylinders. Together with the ideas used in [1], we hope it will help to give a further improvement on the upper bound.

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